ential equations for the eigenvectors, Eq. (5). Two cases are considered; distinct and nondistinct eigenvalues.

When $\lambda_i \neq \lambda_j$, we evaluate the $i-j$th element of Eq. (7), and obtain

$$\Gamma_{ij} = \mathbf{X}'_i\dot{C}\mathbf{X}_j/(\lambda_j - \lambda_i),\ i \neq j, \lambda_i \neq \lambda_j \tag{9}$$

The case of multiple roots, however, requires some discussion.

Recall that if the eigenvectors $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+k}$ correspond to the principal value λ_i, then they satisfy

$$(C - \lambda_i E)\mathbf{X}_j = 0,\ j = i, i+1, \ldots, i+k \tag{10}$$

Therefore,

$$(C - \lambda_i E)\dot{\mathbf{X}}_j = (\dot{\lambda}_i E - \dot{C})\mathbf{X}_j,\ j = i, \ldots, i+k \tag{11}$$

But, from Eq. (5) we have

$$\dot{\mathbf{X}}_j = \sum_{m=1}^{n} \Gamma_{mj}\mathbf{X}_m \tag{12}$$

and substituting this into Eq. (11) results in

$$\sum_{m=1}^{n} \Gamma_{mj}(\lambda_m - \lambda_i)\mathbf{X}_m = (\dot{\lambda}_i E - \dot{C})\mathbf{X}_j \tag{13}$$

Since the eigenvectors $\mathbf{X}_m$ are orthonormal, we must have

$$\begin{aligned}\Gamma_{mj}(\lambda_m - \lambda_i) &= \dot{\lambda}_i\mathbf{X}'_m\mathbf{X}_j - \mathbf{X}'_m\dot{C}\mathbf{X}_j \\ &= \mathbf{X}'_i\dot{C}\mathbf{X}_i\delta_{mj} - \mathbf{X}'_m\dot{C}\mathbf{X}_j\end{aligned} \tag{14}$$

where δ_{mj} is the Kronecker delta. Because $\lambda_m = \lambda_i$ for $m = i, i+1, \ldots, i+k$, the left side of Eq. (14) is zero, independent of Γ_{mj}.‡ Although, we must of necessity have $\Gamma_{jj} = 0$ (because $\mathbf{X}_j \perp \dot{\mathbf{X}}_j$), there seems to be no constraints on the Γ_{mj} ($m \neq j$; $m,j = i, i+1, \ldots, i+k$). This shows that any component vector of $\dot{\mathbf{X}}_j$ which lies in the null space of $C - \lambda_i E$ is annihilated, and because of this it seems quite reasonable to require that $\dot{\mathbf{X}}_j$ be orthogonal to this subspace; i.e., $\dot{\mathbf{X}}_j$ is to be orthogonal to the space generated by $\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+k}$. This convention allows us to write

$$\Gamma_{ij} = 0 \text{ when } \lambda_i = \lambda_j \tag{15}$$

for the case of multiple eigenvalues.

Equations (6, 9, and 14) define Γ and Eq. (5) defines the differential equations governing the propagation of the eigenvectors.

Appendix: Summary of Eigenvalue/Eigenvector Propagation Equations

$$\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n),\ \Lambda = (\lambda_1, \ldots, \lambda_n) \tag{A1}$$

Remark: $C\mathbf{X}_i = \lambda_i\mathbf{X}_i$, $\mathbf{XX}' = E$; if C_0 is diagonal, then one may take $\mathbf{X} = E$ and $\Lambda = C_0$.

$$\dot{C} = A\mathbf{X}\Lambda\mathbf{X}' + (A\mathbf{X}\Lambda\mathbf{X}')' + Q \tag{A2}$$

Remark: $\dot{C}$ is not integrated, but is used to evaluate $\dot{\Lambda}$ and Γ. Thus,

$$\dot{\lambda}_i = \mathbf{X}'_i\dot{C}\mathbf{X}_i \tag{A3}$$

$$\Gamma + \Gamma' = 0 \tag{A4.1}$$

and

$$\Gamma_{ij} = \begin{cases}\mathbf{X}'_i\dot{C}\mathbf{X}'_j/(\lambda_j - \lambda_i) & \text{when } j > i \text{ and } \lambda_j \neq \lambda_i \\ 0 & \text{when } j \geq i \text{ and } \lambda_j = \lambda_i\end{cases} \tag{A4.2}$$

$$\dot{\mathbf{X}} = \mathbf{X}\Gamma \tag{A5}$$

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‡ The right side is, of course, zero also.

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Electron Beam Measurements of Rotational Temperatures with Vibrational Temperatures Greater Than 800°K

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Nomenclature

B_e, α_e = spectroscopic constants defined in Ref. 7
c = speed of light
$E_{v''_1}$ = characteristic energy of vibrational energy level v''_l
$G(K', T_R)$ = defined in Ref. 1
h = Planck's constant
$(I_{K', K_2''})_{v', v_2''}$ = intensity of rotational line emission
I_o = reference intensity
K = rotational quantum number
k = Boltzmann constant
$N_{K'}$ = steady-state number density population of rotational energy state
P_{v', v''_1} = band strength
Q_r = rotational state sum
Q_v = vibrational state sum
T_R = rotational temperature
T_v = vibrational temperature
v = vibrational quantum number
X_1, X_2, X_3 = consts
ν = wave number

Superscripts

$(\ )'$ = quantities for excited electronic state $N_2{}^+B^2\Sigma_u{}^+$
$(\ )''$ = quantities for ground state of neutral nitrogen $N_2X^1\Sigma_g{}^+$ or ionized nitrogen $N_2{}^+X^2\Sigma_g{}^+$

Subscripts

$(\ )_1$ = ground state of neutral nitrogen $N_2X^1\Sigma_u{}^+$
$(\ )_2$ = ground state of ionized nitrogen $N_2{}^+X^2\Sigma_g{}^+$

THE electron beam technique has been used for measuring the molecular rotational and vibrational temperatures T_R and T_v, respectively, of highly expanded low-density flows of air and nitrogen. The original formulation of the theoretical model and experimental verification for T_R and

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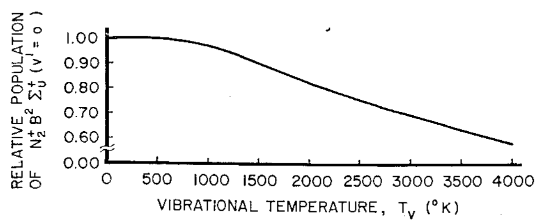

Fig. 1 Relative population variation of the zero vibrational level of $N_2{}^+B^2\Sigma u^+$ with vibrational temperature.

T_v measurements with the electron beam technique were performed by Muntz.[1] The theoretical development for T_R measurements was divided into two cases. The first case was for $T_v < 800°K$ and the second was for $T_v \geq 800°K$. The theoretical expression for the first case is

$$\ln \left\{ \frac{(I_{K',K''_2})v',v''_2/I_0}{(K' + K''_2 + 1)[G(K',T_R)]\nu^4} \right\} = - \frac{B_{v''_1}K'(K'+1)hc}{kT_R} + X_1 \quad (1)$$

and the second case is described by

$$\ln \left\{ \frac{(I_{K',K''_2})_{v,v''_2}/I_0}{(K' + K''_2 + 1)[G(K',T_R)]\nu^4} \right\} = - \frac{B''_{\text{eff}}K'(K'+1)hc}{kT_R} + X_2 \quad (2)$$

The difference between Eqs. (1) and (2) is contained in the rotator constants $B_{v''_1}$ and B''_{eff}. The B''_{eff} constant is a weighted average which may be obtained as described in Ref. 2. The $B_{v''_1}$ term is calculated from

$$B_{v''_1} \approx B_e - \alpha_e(v''_1 + \tfrac{1}{2}) \quad (3)$$

where $v''_1 = 0$ for the case of $T_v < 800°K$.

The situation often exists in high-enthalpy wind tunnels where the freestream T_v is greater than 800°K. It is in principle necessary to use Eq. (2) to obtain T_R. The purpose of this Note is to show that a reformulation of Muntz's theoretical expression permits the determination of T_R from a single expression which is applicable to the low- and high-T_v cases. The reformulated expression is applied to electron beam data obtained from several wind tunnels where T_v was greater than 800°K. The results of T_R calculations with the reformulated expression are compared with the results obtained from Eq. (1). The results support the conclusions of Petrie[3] that Eq. (1) may be applied for a wide range of T_v conditions without introducing significant error in the resultant T_R values. The reasons for this agreement between results using the reformulated expression and Eq. (1) are discussed.

The fundamental physical concepts of the electron beam technique for measuring rotational and vibrational temperatures of molecular nitrogen has been discussed by a number of authors.[1,4–6] When fast electrons pass through low-density air or nitrogen, gas fluorescence is observed which is the result of inelastic collisions of the electrons with the gas molecules. A major source of observed radiation is the first negative system of nitrogen. The (0,0) band of the first negative system of $N_2{}^+$ is the most intense and is the band generally used for measuring T_R. The discussion to follow will be applicable to any band of the first negative system but the results shown were obtained from (0,0) band data.

The ionized and excited state of the first negative system of $N_2{}^+$, $N_2{}^+B^2\Sigma_u{}^+$, is populated as a result of inelastic collisions between the fast primary electrons and the ground state neutral molecules, $N_2X^1\Sigma_g{}^+$. The observed fluorescence is due to the spontaneous transition from $N_2{}^+B^2\Sigma_u{}^+$ to the ground ionized state $N_2{}^+X^2\Sigma_g{}^+$. The theoretical problem is to predict the population distribution of the molecules in the rotational and vibrational energy states of $N_2{}^+B^2\Sigma_u{}^+$ as a function of T_R and T_v, respectively. In order to describe the population distribution of the molecules among the rotational energy states of $N_2{}^+B^2\Sigma_u{}^+$, Muntz[1] assumed that the optical selection rules applied in the excitation process, i.e., $\Delta K = \pm 1$. With this assumption the population of a rotational energy state $N_{K'}$ is proportional to

$$K'e^{-E_{K'-1}/kT_R} + (K'+1)e^{-E_{K'+1}/kT_R} \quad (4)$$

where $E_{K'-1}$ and $E_{K'+1}$ are rotational energy terms that are functions of the rotational quantum state K' and vibrational quantum state v''_1. These terms are defined and may be evaluated in accordance with Ref. 7. By appropriate grouping of the spectroscopic terms of $E_{K'-1}$ and $E_{K'+1}$ expression (4) may be written as

$$[F(K',v''_1,T_R)]e^{-K'(K'+1)B_ehc/kT_R} \quad (5)$$

The $[F(K',v''_1,T_R)]$ term is essentially equivalent to the $[G(K',T_R)]$ term defined by Muntz[1] except the vibrational quantum state dependence has been retained.

The final equation that can be used for determining the rotational temperature is

$$\ln\left\{ [(I_{K',K''_2})v',v''_2/I_0] \Big/ \frac{K'}{2K'+1}\, \nu^4 \sum_{v''_1} \times \left[\frac{P_{v,v''_1} F(K',v''_1,T_R) e^{-E_{v''_1}/kT_v}}{Q_rQ_v} \right] \right\} = - \frac{B_eK'(K'+1)hc}{kT_R} + X_3 \quad (6)$$

This equation must be solved through an iteration process since the $[F]$ term is T_R dependent. Of course, Eqs. (1) and (2) are also solved through an iteration process because of the T_R dependence of the $[G]$ term.

The differences between Eqs. (1) and (6) are 1) B_e, rotational constant which is known, is used instead of $B_{v''_1}$ or B''_{eff}, 2) T_v is now used explicitly in the equation, and 3) a summation over the vibrational quantum states of $N_2X^1\Sigma^+_g$ is required. The upper limit of the summation is determined by the maximum T_v to be considered.

The results of the application of Eq. (6) to various test conditions were examined and compared with T_R values obtained from Eq. (1). The comparison is shown in Table 1. The T_R value obtained from Eq. (1) agrees well with those values obtained from Eq. (6). This indicates that Eq. (1) is applicable over a broad range of T_v conditions for transitions from $N_2{}^+B^2\Sigma_u{}^+(v' = 0)$.

The reason for this agreement can be understood if the relative population of the various rotational energy states of the $N_2{}^+B^2\Sigma_u{}^+(v' = 0)$ is determined. The relative population is calculated from

$$N_{K'}/\sum_{K'} N_{K'} \quad (7)$$

where

$$N_{K'} \propto \sum_{v''_1} \frac{P_{v',v\ 1}(A)\, e^{-E_{v''_1}/kT_v}}{Q_vQ_r} \quad (8)$$

Table 1 Comparison of calculated rotational temperatures using Eqs. (1) and (6)

T_R, °K using Eq. (1)	T_R, °K using Eq. (6)	T_v, °K used in Eq. (6)
39[a]	39	1000
177	177	2700
1021[c]	1018	1025

[a] Test conducted in Langley Research Center hypersonic nitrogen tunnel.[8]
[b] Test conducted in Langley Research Center 1-ft hypersonic tunnel,[4] data supplied by D. I. Sebacher.
[c] Test conducted in static gas.[6]

The quantity (A) is defined by expression (4). As an example of the results obtained from expression (7), a set of calculations was made for the condition of $T_R = 177°K$ and for T_v values of 177° and 2700°K. The results are given in Table 2.

It is noted that there is very little change in the relative population of the excited ionized state of N_2^+ for each T_v considered. There is a slight population shift from the lower quantum states to the upper quantum states for the 2700°K T_v case. Since a relatively large change in the T_v does not significantly change the relative population distribution of the rotational energy states of the zero vibrational energy level of $N_2^+B^2\Sigma_u^+$, only a small difference will exist between the results of Eqs. (1) and (6). Of course, the total population of the zero vibrational energy level of $N_2^+B^2\Sigma^+_u$ does change with T_v as shown in Fig. 1.

Muntz[1,2] noted that the rotational energy-state population distribution for $v' = 0$ energy level of $N_2^+B^2\Sigma_u^+$ was not a significantly strong function of T_v because of the particular distribution of Franck-Condon factors. This point was examined further by comparing the relative population variation of $v' = 0$, 1, and 2 of $N_2^+B^2\Sigma_u^+$ for $T_R = 25°K$ and $T_v = 25°K$ and 4000°K. The increase of T_v from 25° to 4000°K resulted in a shift of population from the lower rotational quantum states to the upper quantum states. The degree of population shift increased with the vibrational quantum number. For example, the relative change in the rotational quantum state $K' = 6$ of $v' = 0$, 1, and 2 was approximately 0.1, 1.8, and 3.9%, respectively.

In summary, Eq. (6) provides a single equation which is applicable for a broad T_v range. A comparison of results obtained with Eqs. (1) and (6) applied to specific cases of transition from the $v' = 0$ level of $N_2^+B^2\Sigma_u^+$ shows that Eq. (1) may be applied for a range of T_v values up to 2700°K without introducing a significant error in the resultant T_R values. It was shown that this was because the population distribution of the $v' = 0$ level of $N_2^+B^2\Sigma_u^+$ was not a strong function of T_v. The application of Eq. (1) to transitions from vibrational energy states other than $v' = 0$ of $N_2^+B^2\Sigma_u^+$ must be done with care since the population shift is greater for higher T_v's.

Table 2 Relative population of the zero vibrational level of $N_2^+B^2\Sigma_u^+$ for 177°K rotational temperature and 177°K and 2700°K vibrational temperature

	T_v	
K'	177°K	2700°K
1	0.06102	0.06100
2	0.04776	0.04775
3	0.12175	0.12173
4	0.06906	0.06904
5	0.14434	0.14433
6	0.07069	0.07068
7	0.13101	0.13101
8	0.05779	0.05779
9	0.09744	0.09746
10	0.03937	0.03938
11	0.06110	0.06112
12	0.02280	0.02281
13	0.03277	0.03279
14	0.01135	0.01136
15	0.01516	0.01517
16	0.00489	0.00489
17	0.00608	0.00609
18	0.00183	0.00183
19	0.00212	0.00213
20	0.00060	0.00060
21	0.00065	0.00065
22	0.00017	0.00017
23	0.00017	0.00017
24	0.00004	0.00004

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Problem of the Annular Plate, Simply Supported and Loaded with an Eccentric Concentrated Force

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Introduction

THE subject of this paper is the determination of the deflection function for the annular plate, simply supported at both edges, and loaded with an eccentric concentrated force. In a previous paper,[1] the problem of the clamped annular plate was discussed. The actual boundary conditions probably vary somewhere between these two cases. According to the Poisson-Kirchoff theory of bending of thin plates,[2] the deflection w is a solution of the equation

$$\nabla^4 w = q/D \qquad (1)$$

where ∇^4 is the biharmonic operator, q is the transverse load intensity, and D the plate flexural rigidity. For a problem of the type considered, the solution can be expressed in the form

$$w = w_S + w_R \qquad (2)$$

Here, w_S represents the singular solution state the w_R the

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